

LATERAL DIFFUSION IN FINITE DIFFERENCE SCHEMES FOR AVALANCHE SIMULATION

Ángel Rodríguez-Cerro¹, Nuria Molner¹, Ignacio García-Fernández² and Rafael J. Martínez Durá¹

¹ LSyM – IRTIC. Universitat de València. Valencia, Spain.

² Departament d'Informàtica. Universitat de València. Valencia, Spain.
email: angel.rodriguez@uv.es

KEYWORDS

Granular Material, Partial Differential Equation, Finite Difference Scheme, Avalanche Simulation

ABSTRACT

Modelling and simulation of granular materials is an active research field with applications in many industry sectors. Topics of interest range from microscopic problems, such as grain segregation to macroscopic phenomena, such as avalanches. The BCRE model (from the authors' names, Bouchaud, Cates, Ravi Prakash and Edwards) is a model that describes the evolution of an avalanche as a thin layer of rolling material over a basis of static grains. The model uses a system of two partial differential equations to compute simulations of the avalanche process. Depending on the numerical scheme used to solve the equations, there are some phenomena appearing in real avalanches which are not properly reproduced in the simulation. In this paper we focus on the lateral expansion of an avalanche that slides down a slope. We analyse two different finite difference schemes that can be used to solve the equations and prove that only one of them can simulate some expansion processes observed in real avalanches. We perform several simulations to show the properties deduced from our analysis.

INTRODUCTION

Granular materials are present in many natural processes, like avalanches of sand or snow. The study of granular systems can play an important role in different industries such as mining, chemical industry or geotechnics. In sectors where manipulation of granular materials is done with heavy machinery it is common that Virtual Reality simulators are used in order to train the operators (García-Fernández et al. (2011)). Our research is motivated by the use of granular systems in Virtual Reality applications that involve granular systems, such as earthmoving scenarios or industrial environments that involve bulk materials (coal, fertilizers or grains). In these simulators, a proper simulation and visualization of avalanches and their evolution is necessary.

We address the problem of finding an appropriate numerical scheme to solve models of avalanches based on

partial differential equations. We compare two finite difference schemes and prove that one of them can simulate the observed shape of real avalanches, while the other one does not propagate according to such observations. In this section we give an introduction to the problem of avalanche modelling and describe the BCRE model of avalanche evolution. In the next section we present our analysis to compare two finite difference schemes used to develop simulations of avalanches. Finally, we present some numerical tests to illustrate our results.

Avalanches in Granular Materials

Granular materials have a property known as the *critical slope angle*. When the slope of the granular system exceeds this angle, a layer of grains starts flowing over the surface of the system, causing an avalanche. The existence of this critical angle is a macroscopic feature resulting from internal friction, and it depends on the properties of the granular material, such as grain size distribution, grain shapes, roughness of surface or material moisture. Most models assume that, during an avalanche, there is a static layer, made of immobile grains, and a rolling layer, made of grains that are sliding down the slope. The interested reader can find further information about the properties of avalanches in granular systems in the works by Savage and Hutter (1991), Duran (1999), Prigozhin and Zaltzman (2001), Aradian et al. (2002), Prigozhin and Zaltzman (2003). Bouchaud et al. (Bouchaud et al. (1994)) proposed an analytical model for an avalanche, the so called BCRE model (from the authors' names, Bouchaud, Cates, Ravi Prakash and Edwards), which is based on a phenomenological description of the process. The BCRE model consists of two equations that describe the evolution of the static layer, $u(x, t)$, and the rolling layer, $v(x, t)$:

$$\begin{aligned} v_t &= \nabla(v \nabla u) - (1 - |\nabla u|)v + f \\ u_t &= (1 - |\nabla u|)v \end{aligned} \quad (1)$$

The equation (1) has been normalized so that the model parameters have value 1 (see the work by Haderl and Kuttler (1999) for more details). The first equation shows two terms, a transport term $\nabla(v \nabla u)$ and a term that accounts for material exchange between layers, $(1 - |\nabla u|)v$. The term f represents a source of gran-

ular material, which allows the introduction of material in the system. This model has been revisited by several authors (Alamino and Prado (2002), Shen (2007), Shen and Zhang (2010), Colombo et al. (2012), Cattani et al. (2012)). The formulation used in this paper is a two dimensional version of the model proposed by Haderl and Kuttler (1999).

According to empirical studies, an avalanche caused at a point in a slope extends forming a characteristic shape. The works by Daerr and Douady (1999) and Daerr (2001) describe two main behaviours. When there is an avalanche formed by a thin rolling layer, it will grow laterally on its way down, forming a triangular shape, whereas avalanches with a thick rolling layer also propagate upwards, eventually causing the whole slope to slide. In both cases, the avalanche not only propagates down the slope but also extends sideways.

In our work we want to simulate avalanches in a slope of granular matter using a discretisation of the system of equations (1). Falcone and Vita (2006) propose a finite difference scheme for the BCRC model and derive a condition for consistency and convergence. However, when using this numerical scheme to simulate an avalanche generated at an isolated point in a slope, it can be observed that avalanche evolution does not show lateral diffusion or uphill avalanche propagation, as described before. As an alternative to this finite difference scheme, we propose the use of the Lax-Friedrichs scheme (Strikwerda (2004)).

In the next section we compare the lateral diffusion of an avalanche when simulated with the scheme proposed by Falcone and Vita (2006) (the FV scheme from now on) and with the standard Lax-Friedrichs finite difference scheme. More precisely, the evolution of a point that is next to the avalanche area is obtained, showing that the FV scheme will not propagate sideways or upwards.

LATERAL AVALANCHE DIFFUSION IN FINITE DIFFERENCE SCHEMES

The goal of the analysis presented in this section is to quantify the lateral and uphill expansion of an avalanche in a slope. We shall focus on a system with an active avalanche, and will analyse the evolution of a point of the finite difference grid that is next to the avalanche, but not affected by it.

We shall consider a regular grid aligned with the X and Y axes, that discretises the spatial domain of the problem. The grid spacing will be denoted as h and the time step for the numerical time integration will be denoted as Δt . The nodes of the grid will be denoted by indexes (i, j) . The value of the discretised solution of (1) at grid point (i, j) and time $t_n = t_0 + n\Delta t$ will be denoted as $(u_{i,j}^n, v_{i,j}^n)$.

Let's consider a grid point (i, j) that has no active avalanche at time t_n . As the thickness of the avalanche

is given by $v(x, t)$ in (1), this is equivalent to $v_{i,j}^n = 0$. And let's consider that this point is in the border of an active avalanche, so that only one of the adjacent nodes has active avalanche. Without loss of generality we shall assume that it is node $(i - 1, j)$, having

$$v_{i-1,j}^n > 0; \quad v_{i+1,j}^n = v_{i,j+1}^n = v_{i,j-1}^n = 0. \quad (2)$$

Moreover, we shall consider that no rolling material is being added at point (i, j) by an external source, thus $f_{i,j} = 0$. Figure 1 shows the situation that is described. In the figure, nodes with double circle represent locations with avalanche, while the ones with only one circle correspond to locations with no avalanche.

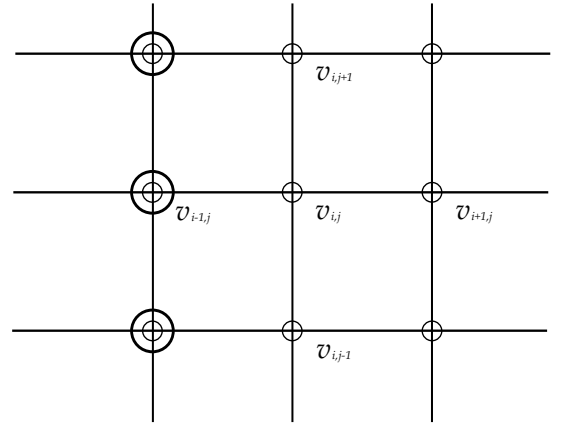


Figure 1: The scenario considered in the analysis. A regular grid is used to discretise the domain. Nodes with double circle indicate the existence of material in the rolling layer. The point of interest is the node (i, j) , which is not affected by the avalanche but is next to a node affected by it.

We are interested in the situation when point (i, j) is located uphill or at the same level than the point with an active avalanche $(i - 1, j)$. Thus, we shall consider that, at point (i, j) , the slope does not go down in the direction from grid point (i, j) to grid point $(i + 1, j)$. That is to say that the gradient of the static layer, $Du_{i,j}(x, t)$, has a positive or null first coordinate,

$$Du_{ij}(x, t) = (a, b); \quad a \geq 0, b \in \mathbb{R}. \quad (3)$$

Next we show that using the FV finite difference scheme the avalanche has no effect on point (i, j) whereas in the Lax-Friedrichs finite difference scheme point (i, j) changes its value.

Analysis of the FV Finite Difference Scheme

The explicit finite difference scheme proposed by Falcone and Vita (2006) to integrate the BCRC model can

be written as

$$\begin{aligned} v_{i,j}^{n+1} &= v_{i,j}^n + \Delta t \left[v_{i,j}^n D^2 u_{i,j}^n + \right. \\ &\quad \left. + \overline{D} v_{i,j}^n \cdot Du_{i,j}^n - (1 - |Du_{i,j}^n|) v_{i,j}^n \right], \quad (4) \\ u_{i,j}^{n+1} &= u_{i,j}^n + \Delta t (1 - |Du_{i,j}^n|) v_{i,j}^n, \end{aligned}$$

where the involved differences are as follows. For any quantity A taking values on the grid, the lateral differences are defined as

$$\begin{aligned} D_x^- A_{i,j} &= \frac{A_{i,j} - A_{i-1,j}}{h}; & D_x^+ A_{i,j} &= \frac{A_{i+1,j} - A_{i,j}}{h}; \\ D_y^- A_{i,j} &= \frac{A_{i,j} - A_{i,j-1}}{h}; & D_y^+ A_{i,j} &= \frac{A_{i,j+1} - A_{i,j}}{h}. \end{aligned} \quad (5)$$

According to Falcone and Vita (2006), the gradient $Du_{i,j}$ is computed choosing the lateral differences that maximize $|Du_{i,j}|$. The gradient of v , denoted by $\overline{D}v_{i,j}$, is approximated by the uphill finite differences with respect to $Du_{i,j}$, defined as

$$\overline{D}_x v_{i,j} \equiv \begin{cases} D_x^+ v_i & \text{if } D_x u_i > 0, D_x^+ u_i > 0, \\ D_x^- v_i & \text{if } D_x u_i < 0, D_x^- u_i < 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6)$$

The definition for $\overline{D}_y v_{i,j}$ is analogous. Finally, $D^2 u_{i,j}$ is approximated using the standard central difference of order two

$$D^2 u_{i,j} = \frac{u_{i,j-1} + u_{i,j+1} - 4u_{i,j} + u_{i-1,j} + u_{i+1,j}}{h^2}.$$

Using the previous definitions, the value of $v_{i,j}^{n+1}$ can now be computed from (4). The first term, involving the second order difference is zero, since $v_{i,j}^n = 0$, and the last term is also null for the same reason, leaving

$$v_{i,j}^{n+1} = \Delta t [\overline{D}v_{i,j}^n \cdot Du_{i,j}^n]. \quad (7)$$

The x component of the gradient $\overline{D}v_{i,j}^n$ is given by (6). According to the condition (3) the gradient of u accomplishes $D_x u_{i,j}^n = a > 0$ and

$$\overline{D}_x v_{i,j}^n \equiv \begin{cases} D_x^+ v_{i,j}^n & \text{if } D_x^+ u_{i,j}^n > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Now, by equation (2), $v_{i,j}^n = v_{i+1,j}^n = 0$ and $D^+ v_{i,j}^n = 0$, leading to

$$\overline{D}_x v_{i,j}^n = 0.$$

Moreover, the y component of $\overline{D}v_{i,j}^n$ is also zero since, again by (2), $v_{i,j-1}^n = v_{i,j}^n = v_{i,j+1}^n = 0$ and

$$D_y^+ v_{i,j}^n = D_y^- v_{i,j}^n = 0. \quad (8)$$

Plugging (8) into (7) it is finally seen that

$$v_{i,j}^{n+1} = 0. \quad (9)$$

Analysis of the Lax-Friedrichs Finite Difference Scheme

The equation for the rolling layer $v(x, t)$ in (1) can be discretised using the Lax-Friedrichs finite difference scheme (Strikwerda (2004)) leading to:

$$v_{i,j}^{n+1} = \overline{v}_{i,j}^n + \Delta t [v_{i,j}^n D^2 u_{i,j}^n + Dv_{i,j}^n \cdot Du_{i,j}^n - (1 - |Du_{i,j}^n|) v_{i,j}^n] \quad (10)$$

In this case, the gradients are computed by central differences

$$D_x v_{i,j}^n = \frac{v_{i+1,j}^n - v_{i-1,j}^n}{2h}; \quad D_y v_{i,j}^n = \frac{v_{i,j+1}^n - v_{i,j-1}^n}{2h}, \quad (11)$$

and the value of $v_{i,j}^n$ is approximated by

$$\overline{v}_{i,j}^n = \frac{v_{i+1,j}^n + v_{i-1,j}^n + v_{i,j+1}^n + v_{i,j-1}^n}{4} \quad (12)$$

From equation (10), and taking into account that $v_{i,j}^n = 0$, we can derive the increment of the rolling layer as

$$v_{i,j}^{n+1} = \overline{v}_{i,j}^n + \Delta t [Dv_{i,j}^n \cdot Du_{i,j}^n] \quad (13)$$

Using the hypotheses that only $v_{i-1,j}^n \neq 0$, expressed by equation (2), and that $D_x u_{i,j}^n = a$, by equation (3), we can develop the dot product in equation (13). Using the central differences (11),

$$\begin{aligned} Dv_{i,j}^n \cdot Du_{i,j}^n &= \frac{(v_{i+1,j}^n - v_{i-1,j}^n)(u_{i+1,j}^n - u_{i-1,j}^n)}{4h^2} + \\ &\quad + \frac{(v_{i,j+1}^n - v_{i,j-1}^n)(u_{i,j+1}^n - u_{i,j-1}^n)}{4h^2} \\ &= -\frac{v_{i-1,j}^n}{2h} \left(\frac{u_{i+1,j}^n - u_{i-1,j}^n}{2h} \right) \\ &= -\frac{v_{i-1,j}^n}{2h} a. \end{aligned} \quad (14)$$

Applying again that only $v_{i-1,j}^n \neq 0$, the approximation to $v_{i,j}^n$ given by (12) results in

$$\overline{v}_{i,j}^n = \frac{v_{i-1,j}^n}{4}. \quad (15)$$

Finally, substituting (14) and (15) into (13), we have that using the Lax-Friedrichs finite difference scheme the value of the rolling layer at point (i, j) will be

$$\begin{aligned} v_{i,j}^{n+1} &= \overline{v}_{i,j}^n + \Delta t [Dv_{i,j}^n \cdot Du_{i,j}^n] = \\ &= v_{i-1,j}^n \left(\frac{1}{4} - a \frac{\Delta t}{2h} \right), \end{aligned} \quad (16)$$

producing a change in the value of $v_{i,j}^n$ in any situation. The only case when (16) vanishes is in the particular situation when $a = \frac{h}{2\Delta t}$. This situation, however, is unlikely to last in time, since the evolution of the avalanche will cause that the situation disappears in very few time steps.

Discussion

In the last two subsections we have proved that, under the assumptions (2) and (3), the finite difference scheme proposed in the work by Falcone and Vita (2006) only propagates an avalanche provoked at a point in the downhill direction of the simulated granular system. Under the same conditions, but using the Lax-Friedrichs finite difference scheme, avalanches expand in all directions, showing an evolution that is more coherent with empirical studies.

Thus, the FV scheme is adequate for situations of massive avalanches, where the lateral avalanche expansion is not relevant, as shown by Falcone and Vita (2006) in their work. Moreover the FV scheme is known to be convergent and stable under certain conditions. However, in situations where the avalanche starts as a more local process and we are interested in the geometry of the avalanche as it evolves, the averaging term in the Lax-Friedrichs scheme leads to better qualitative results.

To the best of our knowledge, there is no convergence and stability analysis for the Lax-Friedrichs scheme applied to this problem. In our numerical tests, described next, we have used the same criteria for both numerical schema, and stability has been achieved in the two methods.

NUMERICAL TESTS

We have simulated an avalanche on a static slope with 45 degrees of inclination. We provoke the avalanche in a bounded region, discretised as a square of 3×3 grid cells, by increasing the amount of material in the rolling layer. The whole system is discretised by a grid of 80×80 cells. Figure 2 shows the evolution of the avalanche, using the finite difference scheme proposed by Falcone and Vita (2006). In the figure, the slope points in the down direction. The figure shows the region occupied by the avalanche after 2.8 seconds (left) and the intermediate states after every 0.2 seconds of simulation time. As predicted, the material flows down the slope, but it does not slide laterally, perpendicular to the main slope, and does not expand uphill.

Figure 3 shows the evolution of the avalanche simulated with the Lax-Friedrichs finite difference scheme. The figure shows a similar representation as the depicted in the previous figure. In this simulation, the avalanche slides down the slope and expands laterally. Moreover, it can be observed that the avalanche also propagates upwards. This reproduces real avalanche behaviour, in which grains located uphill progressively tumble down because of loss of support, as it is described in Daerr and Douady (1999).

Recalling the motivation of this work, we have implemented a graphical simulation, in order to show the application of the results in the context of a Virtual Reality environment. In Figure 4 we show an OpenGL visualiza-

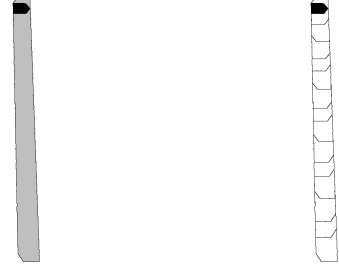


Figure 2: Avalanche propagation using the FV finite difference scheme. Left image shows the silhouette of avalanche after 2.8 seconds from the start. Right image shows the expansion of the avalanche in intervals of 0.2 seconds. The black spot indicates the point where the avalanche was generated.

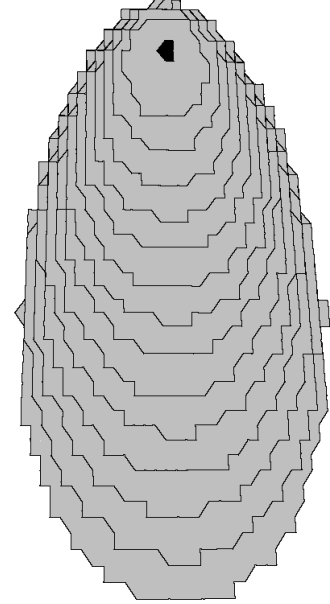


Figure 3: Avalanche propagation using the Lax-Friedrichs scheme. The image shows the silhouette of avalanche after 2.8 seconds from the start. The lines indicated the expansion of the avalanche in intervals of 0.20 seconds.

tion of the model using both numerical finite difference schemes. The visualization is done by means of texture advection as described by Rodriguez-Cerro et al. (2015) on a triangle mesh that is built upon the discretisation of the BCRC model.

Since computation time is relevant in the models used in Virtual Reality simulations, it is noteworthy that the

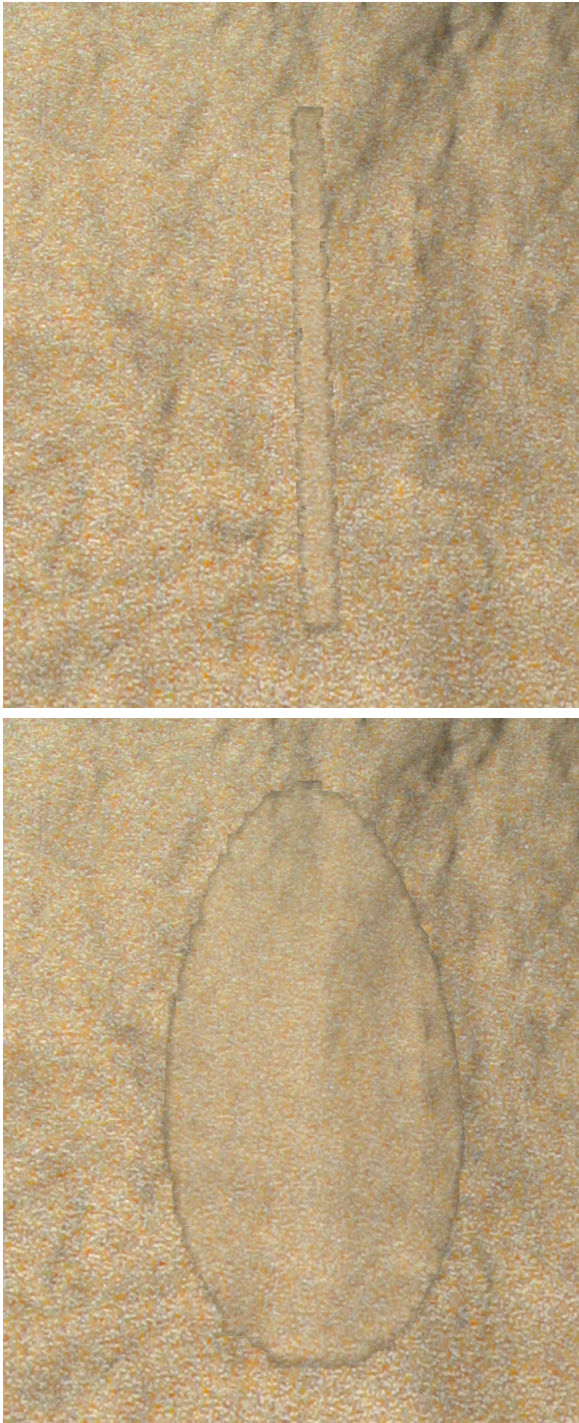


Figure 4: An OpenGL visualization of an avalanche simulated using the FV finite difference scheme (top) and the Lax-Friedrichs finite difference scheme (bottom).

two finite difference schemes are explicit schemes and require a number of computations of the same order of magnitude.

CONCLUSION AND FUTURE WORK

We have addressed the simulation of avalanches in a slope of granular matter using two finite difference discretisations of the BCRE model. We have proved that the scheme proposed by Falcone and Vita (2006) does not propagate laterally and upwards on the slope. This behaviour is not consistent with the behaviour of avalanches observed by empirical studies (Daerr and Douady (1999)). We have also demonstrated that the Lax-Friedrich finite difference scheme, applied to the BCRE model, produces the formation of avalanches that have a shape more consistent with the aforementioned empirical observations. An analysis of stability and convergence of the Lax-Friedrichs scheme has yet to be done for this particular problem.

REFERENCES

- Alamino R.C. and Prado C.P., 2002. *Some considerations on the Bouchaud-Cates-Ravi-Edwards model for granular flow*. *Physica A: Statistical Mechanics and its Applications*, 308, no. 1, 148–160.
- Aradian A.; lie Raphael; and deGennes P.G., 2002. *Surface flows of granular materials: a short introduction to some recent models*. *Comptes Rendus Physique*, 3, no. 2, 187 – 196.
- Bouchaud J.P.; Cates M.; Ravi Prakash J.; and Edwards S., 1994. *A model for the dynamics of sandpile surfaces*. *Journal de Physique I*, 4, no. 10, 1383–1410.
- Cattani A.; Colombo R.; and Guerra G., 2012. *A hyperbolic model for granular flow*. *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik*, 92, no. 1, 72–88.
- Colombo R.M.; Guerra G.; and Monti F., 2012. *Modelling the dynamics of granular matter*. *IMA Journal of Applied Mathematics*, 77, no. 2, 140–156.
- Daerr A., 2001. *Dynamical equilibrium of avalanches on a rough plane*. *Physics of Fluids (1994-present)*, 13, no. 7, 2115–2124.
- Daerr A. and Douady S., 1999. *Two types of avalanche behaviour in granular media*. *Nature*, 399, no. 6733, 241–243.
- Duran J., 1999. *Sands, Powders and Grains*. Springer.
- Falcone M. and Vita S.F., 2006. *A finite-difference approximation of a two-layer system for growing sand-piles*. *SIAM Journal on Scientific Computing*, 28, no. 3, 1120–1132.
- García-Fernández I.; Pla-Castells M.; Gamón M.A.; and Martínez-Durá R.J., 2011. *New developments*

in simulation-based harbour crane training. International Journal of Simulation and Process Modelling, 6, no. 4, 274–287.

Hadeler K.P. and Kuttler C., 1999. *Granular Matter in a Silo. Granular Matter*, 3, 193–197.

Prigozhin L. and Zaltzman B., 2001. *Two continuous models for the dynamics of sandpile surfaces. Physical Review E*, 63, no. 4, 041505.

Prigozhin L. and Zaltzman B., 2003. *On the approximation of the dynamics of sandpile surfaces. Portugaliae Mathematica Nova Serie*, 60, no. 2, 127–137.

Rodriguez-Cerro A.; Garcia-Fernandez I.; Martinez-Dura R.J.; and Pla-Castells M., 2015. *Texture advection on discontinuous flows. The Visual Computer*, in press.

Savage S. and Hutter K., 1991. *The dynamics of avalanches of granular materials from initiation to runout. Part I: Analysis. Acta Mechanica*, 86, no. 1-4, 201–223.

Shen W., 2007. *On the Shape of Avalanches. J Math Anal Appl*, 828–838.

Shen W. and Zhang T., 2010. *Erosion Profile by a Global Model for Granular Flow*.

Strikwerda J., 2004. *Finite Difference Schemes and Partial Differential Equations*. SIAM: Society for Industrial and Applied Mathematics, 2 ed.